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EFFECTS OF NON-PROPAGATING PLATE WAVES ON DYNAMICAL STRESS CONCENTRATIONS

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Abstract—The concentration of stress around a circular hole in an infinite elastic plate during passage of a plane extensional wave is studied within the framework of an approximate theory which takes into account the effect of coupling between extensional symmetric thickness-stretch and symmetric thickness-shear deformations. Of particular interest is the very large stress concentrations that are achieved for certain combinations of plate thickness, hole diameter and incident wavelength. The source of these large stress concentrations is seen to lie in the presence of a vibration of large magnitude which is confined to a region in the immediate vicinity of the hole.

INTRODUCTION

In a previous paper [1] on the effects of non-propagating plate waves (i.e. waves with complex wavelengths) on the dynamical state of stress in an elastic plate in which there is a circular hole, an approximate theory [2] which incorporates such waves was used to study the reflection from such a hole of the single extensional plate wave which is propagating at all frequencies. The solution obtained was investigated to ascertain under what conditions it would produce the same results as would generalized plane stress theory for the reflected extensional and face shear waves which have real wavelengths. Generalized plane stress theory, of course, does not contain any non-propagating plate waves. The results showed that agreement would be achieved in the limit of zero plate thickness/incident wavelength ratio provided the ratio of plate thickness to hole diameter also goes to zero.

In the present paper the state of stress in the immediate vicinity of the hole during passage of plane extensional waves is obtained using the second order equations and is discussed. In particular, interest is centered on any concentration of stress that may arise due to the presence of the hole. The problem studied here has previously been investigated within the framework of generalized plane stress theory by Pao [3] who found a maximum increase in stress concentration factor (i.e. defined as ratio of maximum value obtained for principal stress resultant with the hole present to maximum value obtained without the hole present) of approximately 10% over that which occurs in the statical case. His results are independent of the thickness of the plate which does not explicitly appear in generalized plane stress theory. It was one of the purposes of this study to investigate the range of validity of Pao's solution by numerically comparing the results for several values of plate thickness. With respect to this, it should be kept in mind that the numerical difference in results predicted by the two theories is not a true measure of the accuracy of either theory and in the absence of experimental verification, experience must be relied upon to conclude that if the numerical difference is small, the error associated with either will be small. In addition to serving as a guide as to the limitations of the generalized plane stress solution, the second order solution contains some information as to the variation of stress across the thickness of the plate whereas the generalized plane stress solution can only give the average stress across the thickness.

Finally, the second order solution allows an investigation of the effects of "edge vibrations", that arise in the vicinity of any traction free boundary, on stress concentrations. The presence of edge vibrations was first noted by Shaw [4] in his experimental investigations of axially symmetric extensional vibrations of barium titanate disks. Shaw's experiments further showed that at a specific frequency these edge vibrations were in resonance with the radial extensional motion. Gazis and Mindlin [5] were able to show that the origin of the edge vibrations resides in the non-propagating plate waves. They accomplished this by using the second order equations used in this report to solve the problem Shaw investigated experimentally and were able to predict the presence of the edge vibrations and of the resonant like behavior of these vibrations at a frequency near, although somewhat lower, than Shaw's measured value. The possibility of obtaining vibrations of large magnitude confined to the region of a hole in a plate introduces the possibility of extremely large stress concentrations. The investigation of this possibility is beyond that of generalized plane stress theory.

Briefly, the results show rather good agreement for the stress resultants as predicted by the two theories in a region limited to small values of plate thickness to hole diameter ratio and of hole diameter to incident wavelength ratio except for ranges of values of these quantities in the immediate vicinity of certain discrete values. The divergence of the stress distribution across the thickness from that of a uniform distribution is also found to be small over the same region as above, except in the vicinity of the same discrete values mentioned above. Finally, it is possible to relate the combinations of plate thickness, hole diameter and incident wavelength at which there are large discrepancies in the results predicted by the two solutions to the presence of an edge resonance as discussed above.

SECOND ORDER EQUATIONS

Extensional motions which are varying harmonically in time are governed, according to the Mindlin–Medick approximation, by the equations

$$\begin{aligned} \nabla^2\phi_j+\xi_j^2\phi_j &= 0 \qquad j = 1,2,3 \\ \mu\nabla^2\psi_1+\rho\omega^2\psi_1 &= 0 \\ \mu\nabla^2\psi_2+\left(\rho k_4^2\omega^2-\frac{15\mu k_2^2}{b^2}\right)\psi_2 &= 0 \end{aligned} \tag{1}$$

In equations (1), the displacement potentials $\varphi_j (j = 1,2,3)$ and $\boldsymbol{\Psi}_j = \psi_j\mathbf{n}$ $(j = 1,2)$ where $\mathbf{n}$ represents a unit vector normal to the plane of the plate are related to the plate displacements $\mathbf{u}^{(0)}$, $u_2^{(1)}$ and $\mathbf{u}^{(2)}$ as defined in Ref. [2], according to

$$\begin{aligned} \mathbf{u}^{(0)} &= \sum_{j=1}^{3} \nabla\phi_j+\nabla \mathbf{x}\boldsymbol{\psi}_1 \\ u_2^{(1)} &= \sum_{j=1}^{3} \alpha_j\phi_j \\ \mathbf{u}^{(2)} &= \sum_{j=1}^{3} \beta_j\nabla\phi_j+\nabla \mathbf{x}\boldsymbol{\psi}_2, \end{aligned} \tag{2}$$

where

$$\alpha_j = \frac{b}{\lambda k_1}[(\lambda+2\mu)\xi_j^2 - \rho\omega^2] \qquad j = 1, 2, 3$$
$$\beta_j = \frac{\alpha_j b}{\left[\dfrac{\pi^2}{20}\dfrac{k_4^2}{k_2^2}\Omega^2 - 3 - \dfrac{E'b^2}{5\mu k_2^2}\xi_j^2\right]} \qquad j = 1, 2, 3 \tag{3}$$

In these equations, ∇ is the two dimensional gradient operator, λ and μ are Lamé's constants, ρ is the mass density, b is the half-plate thickness, ω is the circular frequency and k_i ($i = 1, 2, 3, 4$) are correction factors which are introduced in order to improve the agreement between the infinite plate frequency spectrum predicted by these equations and that predicted by the full three dimensional theory. In addition,

$$\Omega = \frac{\omega}{\omega_s}; \qquad \omega_s = \frac{\pi}{2b}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \tag{4}$$

and

$$E' = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \tag{5}$$

Furthermore, ξ_j ($j = 1, 2, 3$), the propagation constants of the three extensional modes, are given by the cubic equation

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{vmatrix} = 0 \tag{6}$$

where

$$\begin{aligned} a_{11} &= K^2 Z^2 - \Omega^2 \\ a_{22} &= \frac{k^2}{3} Z^2 + \frac{4K^2 k_1^2}{\pi^2} - \frac{k_3^2}{3}\Omega^2 \\ a_{33} &= \frac{E'}{3} Z^2 + \frac{12 k_2^2}{\pi^2} - \frac{k_4^2}{5}\Omega^2 \\ a_{12} &= \frac{2(K^2-2)k_1}{\pi} Z \\ a_{23} &= -\frac{2k_2^2}{\pi} Z \end{aligned} \tag{7}$$

and

$$\begin{aligned} Z &= \frac{2\xi b}{\pi} \\ K^2 &= \frac{\lambda+2\mu}{\mu} = \frac{2(1-\nu)}{(1-2\nu)} \end{aligned} \tag{8}$$

In equations (8), ν is Poisson's ratio.

The state of stress at a point may be expressed in terms of quantities invariant under a transformation in the plane of the plate by means of two dyadics $\underline{\underline{\tau}}^{(0)}$ and $\underline{\underline{\tau}}^{(2)}$, one vector $\tau_2^{(1)}$ and one scalar $\tau_{22}^{(0)}$. The physical interpretation to be given to these quantities is evident from [2]. They are related to the displacements according to

$$\begin{aligned}
\underline{\underline{\tau}}^{(0)} &= 2\left[\lambda\left(\nabla\cdot\mathbf{u}^{(0)}+\frac{k_1}{b}u_2^{(1)}\right)\underline{\underline{I}}+\mu(\nabla\mathbf{u}^{(0)}+\mathbf{u}^{(0)}\nabla)\right] \\
\underline{\underline{\tau}}^{(2)} &= \tfrac{2}{5}[E'\nu(\nabla\cdot\mathbf{u}^{(2)})\underline{\underline{I}}+\mu(\nabla\mathbf{u}^{(2)}+\mathbf{u}^{(2)}\nabla)] \\
\tau_2^{(1)} &= \frac{2\mu k_2^2}{b}\mathbf{u}^{(2)}+\frac{2\mu k_2^2}{3}\nabla u_2^{(1)} \\
\tau_{22}^{(0)} &= 2\left[\frac{(\lambda+2\mu)k_1^2}{b}u_2^{(1)}+\lambda k_1\nabla\cdot\mathbf{u}^{(0)}\right]
\end{aligned} \tag{9}$$

In equation (9), $\underline{\underline{I}}$ represents the Idemfactor.

INCIDENT AND REFLECTIVE WAVES

Following Pao [3], consider the case of an incident train of plane extensional waves which propagate in the lowest plate mode in the positive x direction. Such a wave train can be represented by a potential $\varphi_1^{(i)}$ of the form:

$$\varphi_1^{(i)} = \varphi_0 \exp i(\xi_1 x-\omega t) \tag{10}$$

where ξ_1 is the propagation constant of the lowest extensional mode. By applying the Fourier–Bessel expansion theorem, $\varphi_1^{(i)}$ can be expressed in polar coordinates (r,θ) as:

$$\varphi_1^{(i)} = \sum_{n=0}^{\infty} \varphi_0\epsilon_n i^n J_n(\xi_1 r)\cos(n\theta)\exp(-i\omega t) \tag{11}$$

in which

$$\begin{aligned}
\epsilon_n &= 1 \qquad \text{if} \quad n = 0 \\
&= 2 \qquad \text{if} \quad n \geqslant 1
\end{aligned}$$

and J_n denotes the Bessel function of the first kind of order n.

Upon striking a traction free boundary at $r = a$, the incident wave will generate outgoing extensional waves and shear waves which are expressed by

$$\begin{aligned}
\varphi_j^{(r)} &= \sum_{n=0}^{\infty} A_{nj}H_n^{(1)}(\xi_j r)\cos(n\theta)\exp(-i\omega t); \qquad j = 1,2,3 \\
\psi_j^{(r)} &= \sum_{n=0}^{\infty} B_{nj}H_n^{(1)}(\delta_j r)\sin(n\theta)\exp(-i\omega t); \qquad j = 1,2
\end{aligned} \tag{12}$$

where $H_n^{(1)}$ denotes the Hankel function of the first kind of order n. The potentials defined by equations (12) will satisfy equations (1) provided the ξ_j's are the propagation constants of the extensional modes and the δ_j's are the propagation constants of the face shear modes given by

$$\delta_1^2 = \frac{\rho\omega^2}{\mu}$$

and (13)

$$\delta_2^2 = \frac{\rho k_4^2 \omega^2}{\mu} - \frac{15k_2^2}{b^2}$$

The coefficients A_{nj} ($j = 1, 2, 3$; $n = 1, \ldots$) and B_{nj} ($j = 1, 2$; $n = 1, \ldots$) are determined by the conditions to be satisfied on the boundary of the circular cavity. For a traction free surface, the appropriate conditions are

$$\tau_{rr}^{(0)}(r = \mathrm{a}) = \tau_{r\theta}^{(0)}(r = \mathrm{a}) = \tau_{2r}^{(1)}(r = \mathrm{a}) = \tau_{rr}^{(2)}(r = \mathrm{a}) = \tau_{r\theta}^{(2)}(r = \mathrm{a}) = 0 \tag{14}$$

where, in the above set of equations, the τ's represent the sums of stresses resulting from incoming and outgoing waves.

Substitution of the potentials given by equations (11) and (12) into equations (2) and the result into equations (9) and then making use of equations (14) and the orthogonality property of $\cos(n\theta)$ over a range of 2π results in a system of five linear algebraic equations for each set of reflected amplitudes. Letting $C_{nj} = A_{nj}/\varphi_0$; $j = 1, 2, 3$ and $C_{nj} = B_{nj-3}/\varphi_0$; $j = 4, 5$, the five equations which arise for a specific term, n, may be expressed as

$$\sum_{j=1}^{5} T_{ij}^{(n)} C_{nj} = F_{ni} \qquad i = 1, 2, 3, 4, 5 \tag{15}$$

where

$$T_{ij}^{(n)} = \left[\frac{2n(n-1)}{S^2} - \Omega^2\right] H_n^{(1)}(Z_j S) + \frac{2}{S} Z_j H_{n+1}^{(1)}(Z_j S) \qquad j = 1, 2, 3$$

$$T_{2j}^{(n)} = \frac{2n}{S}\left[Z_j H_{n+1}^{(1)}(Z_j S) - \frac{n-1}{S} H_n^{(1)}(Z_j S)\right] \qquad j = 1, 2, 3$$

$$T_{3j}^{(n)} = \frac{\beta_j}{1-\nu}\left\{\left[\frac{(1-\nu)n(n-1)}{S^2} - Z_j^2\right] H_n^{(1)}(Z_j S) + \frac{1-\nu}{S} Z_j H_{n+1}^{(1)}(Z_j S)\right\} \qquad j = 1, 2, 3$$

$$T_{4j}^{(n)} = \beta_j T_{2j} \qquad j = 1, 2, 3$$

$$T_{5j}^{(n)} = \left(\beta_j + \frac{\bar{\alpha}_j}{3}\right)\left[\frac{n}{S} H_n^{(1)}(Z_j S) - Z_j H_{n+1}^{(1)}(Z_j S)\right] \qquad j = 1, 2, 3$$

$$T_{14}^{(n)} = \frac{2n}{S}\left[Z_4 H_{n+1}^{(1)}(Z_4 S) - \frac{n-1}{S} H_n^{(1)}(Z_4 S)\right] \tag{16}$$

$$T_{24}^{(n)} = \left[\frac{2n(n-1)}{S^2} - Z_4^2\right] H_n^{(1)}(Z_4 S) + \frac{2}{S} Z_4 H_{n+1}^{(1)}(Z_4 S)$$

$$T_{35}^{(n)} = \frac{n}{S}\left[Z_5 H_n^{(1)}(Z_5 S) - \frac{n-1}{S} H_n^{(1)}(Z_5 S)\right]$$

$$T_{45}^{(n)} = \left[\frac{2n(n-1)}{S^2} - Z_5^2\right] H_n^{(1)}(Z_5 S) + \frac{2}{S} Z_5 H_{n+1}^{(1)}(Z_5 S)$$

$$T_{55}^{(n)} = \frac{n}{S} H_n^{(1)}(Z_5 S)$$

$$T_{15}^{(n)} = T_{25}^{(n)} = T_{34}^{(n)} = T_{44}^{(n)} = T_{54}^{(n)} = 0$$

and the F_{ni}'s are given by

$$F_{ni} = -\epsilon_n i^n \operatorname{Re}[T_{i1}^{(n)}] \qquad i = 1, 2, 3, 4, 5 \tag{17}$$

In equations (16) and (17) the following non-dimensional quantities have been introduced, in addition to those given previously.

$$\begin{aligned} Z_{4,5} &= 2b\delta_{1,2}/\pi \\ \bar{\alpha}_j &= b\alpha_j \\ S &= \pi a/2b \end{aligned} \tag{18}$$

The generalized plane stress solution to the same problem may be obtained by setting $C_2 = C_3 = C_5 = 0$, and using the first two algebraic equations, with Z_1 taken as its limiting form as Ω goes to zero, to determine C_1 and C_4. It is not difficult to show that the generalized plane stress solution is independent of the ratio $S = \pi a/2b$.

PRINCIPAL STRESSES AND STRESS CONCENTRATIONS

The maximum principal stress resultant due to the incident wave alone is given by

$$\tau_x^{(0)} = \frac{-4\varphi_0}{1-\nu} \tag{19}$$

Since $\tau_{r\theta}^{(0)} = \tau_{r\theta}^{(2)} = 0$ at $r = a$, $\tau_{rr}^{(0)} = 0$ and $\tau_{\theta\theta}^{(0)}$ are the principal stress resultants and $\tau_{rr}^{(2)} = 0$ and $\tau_{\theta\theta}^{(2)}$ are the principal stress double moments at the boundary. Making use of the equations defining $\tau_{\theta\theta}^{(0)}$ and $\tau_{\theta\theta}^{(2)}$, it follows that the incident and reflected waves give rise to the following values at the boundary $r = a$

$$\begin{aligned} \tau_\theta^{(0)} &= \frac{\pi^2\mu\varphi_0}{2b^2} \sum_{n=0}^{\infty} \left(G_{n0} + \sum_{j=1}^{3} A_{nj} G_{nj} + B_{n0} G_{n4} \right) \cos(n\theta) \exp(-i\omega t) \\ \tau_\theta^{(2)} &= \frac{-\pi^2\mu\varphi_0}{5(1-\nu)b^2} \sum_{n=0}^{\infty} \left(K_{no} + \sum_{j=1}^{3} A_{nj} K_{nj} + B_{n2} K_{n5} \right) \cos(n\theta) \exp(-i\omega t) \end{aligned} \tag{20}$$

where

$$\begin{aligned} G_{n0} &= \epsilon_n i^n \left\{ \left[2Z_1^2 - \Omega^2 - \frac{2n(n-1)}{S^2} \right] J_n(Z_1 S) - \frac{2Z_1}{S} J_{n+1}(Z_1 S) \right\} \\ G_{nj} &= \left[2Z_j^2 - \Omega^2 - \frac{2n(n-1)}{S^2} \right] H_n(Z_1 S) - \frac{2Z_j}{S} H_{n+1}(Z_j S); \qquad j = 1, 2, 3 \\ G_{n4} &= \frac{2n}{S} \left[\frac{(n-1)}{S} H_n(Z_4 S) - Z_4 H_{n+1}(Z_4 S) \right] \\ K_{n0} &= \epsilon_n i^n \beta_1 \left\{ \left[(1-\nu)\frac{n(n-1)}{S^2} + \nu Z_1^2 \right] J_n(Z_1 S) + (1-\nu)\frac{Z_1}{S} J_{n+1}(Z_1 S) \right\} \\ K_{nj} &= \beta_j \left\{ \left[(1-\nu)\frac{n(n-1)}{S^2} + \nu Z_j^2 \right] H_n(Z_j S) + (1-\nu)\frac{Z_1}{S} H_{n+1}(Z_j S) \right\} \\ K_{n5} &= (1-\nu)\frac{n}{S} \left[Z_5 H_{n+1}(Z_j S) - \frac{(n-1)}{S} H_n(Z_j S) \right] \end{aligned} \tag{21}$$

Neglecting the harmonic time variation, Pao [3] defined the ratio $\tau_\theta^{(0)}/\tau_x^{(0)}$ as the stress concentration factor. Since the average stress across the thickness of the plate is all that is included in generalized plane stress theory such a definition appears to be the only meaningful one. The second order theory, on the other hand, contains some information on the manner in which the stress varies across the plate thickness by giving not only the average stress but also the average of the second moment of the stress distribution taken about the median plane of the plate. It would therefore appear to be more meaningful, in the present case, to attempt to use $\tau_\theta^{(0)}$ and $\tau_\theta^{(2)}$ to construct an approximate stress distribution across the plate and then use the maximum value so found in a definition of the stress concentration factor. One way of doing this would be to proceed in a manner analogous to the derivation of the second order equations and assume that the stress distribution across the thickness is expandable in terms of Legendre polynomials in the thickness coordinate. Then, $\tau_\theta^{(0)}$ and $\tau_\theta^{(2)}$ are simply the coefficients of the zero-th order and second order polynomials, respectively. If we now assume that all other coefficients are zero, an approximation to the stress distribution is obtained. Unfortunately, as explained in Ref. [2], the second order Legendre polynomial does not represent a good approximation to either the distribution of the thickness-stretch or the thickness-shear modes of the exact theory. Indeed, it was the discrepancy in just these approximations which led to the incorporation of the correction factors of the second order theory.

For this reason, no attempt has been made to obtain the maximum stress across the thickness and the stress concentration factor is defined as before, namely $\tau_\theta^{(0)}/\tau_x^{(0)}$. The ratio $\tau_\theta^{(1)}/\tau_\theta^{(0)}$ is taken to give a measure of the average difference between the actual stress distribution and a uniform stress distribution.

RESULTS AND DISCUSSION

By taking the appropriate limits, it is possible to show that the above solution reduces to the generalized plane stress solution [3] as the plate thickness-incident wavelength ratio approaches zero provided that the plate thickness-hole diameter ratio also approaches zero at an equal to or faster rate. It is also possible to show that the solution reduces to that of a plane wave reflecting from a plane boundary [5] in the single limit of the plate thickness/hole diameter ratio approaching zero.

Numerical results were obtained from the above solution by means of an IBM 7040. As explained previously, [2], a specified value for Poisson's ratio, taken here to equal 0·30, determines the compensating factors k_i $(i = 1, 2, 3, 4)$. For an assumed value of the non-dimensionalized frequency, Ω, the propagation constants for the three extensional waves can be obtained from the cubic equation (6) while those for the two face shearing waves can be calculated directly by means of equation (13). With these results and a specified value for plate thickness-hole diameter ratio, equations (16) and (17) determine $T_{ij}^{(n)}$ $(i = 1, \ldots 5; j = 1, \ldots 5)$ and F_{ni} $(i = 1, \ldots 5)$ which in turn determine the amplitudes of the waves generated at the boundary of the hole by a propagating plane extensional wave of the assumed frequency. Finally, knowing the amplitudes of the waves generated by the cavity allows the principal stresses and principal double moments at the cavity to be obtained by direct calculation according to equations (20).

Following the above procedure a table was calculated relating Ω to $\tau_\theta^{(0)}$ and $\tau_\theta^{(2)}$ with a/b as a parameter. All intermediate calculations were carried out to an accuracy of 10^{-8}

and the series defining $\tau_\theta^{(0)}$ and $\tau_\theta^{(2)}$ were truncated when the ratio of the last term retained to the sum was less than 10^{-4}. This required retaining a maximum of 16 terms of the series. The tabulated results are summarized in Figs. 1–6.

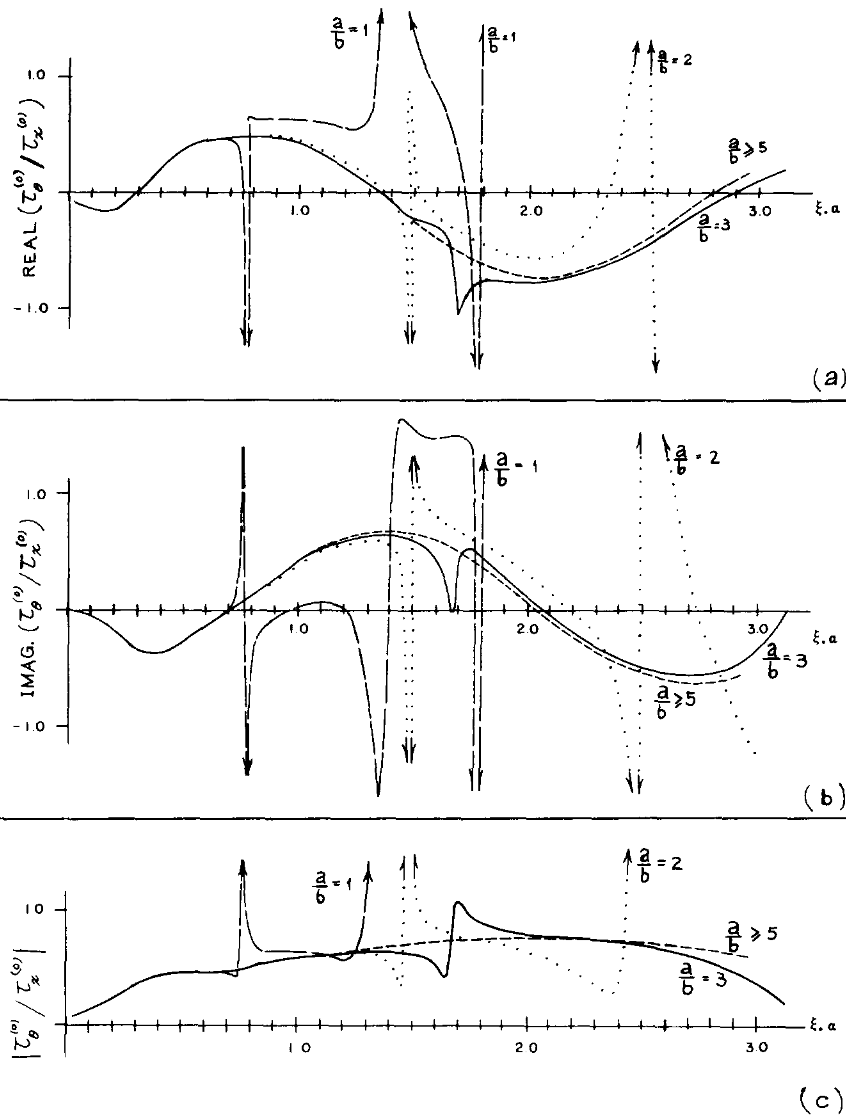

FIG. 1. Stress concentration factor, $\theta = 0$. (a) real part, (b) imaginary part, (c) magnitude.

Figure 1 shows the stress concentration factor $\tau_\theta^{(0)}/\tau_x^{(0)}$ at $\theta = 0$ as a function of the hole diameter/incident wavelength ratio (ξ, a) for various values of hole diameter/plate thickness ratio (a/b). As noted in the figure, the graphs for $a/b \geqslant 5$, are all identical, for the scale chosen, over the entire range of values for ξ, a that is shown. This agrees with the generalized plane stress solution which predicts results to be independent of a/b. A comparison with the results presented by Pao [3] shows no noticeable difference between the results predicted here for $a/b \geqslant 5$ and those predicted by the generalized plane stress theory. Again this statement is valid only for the range of values shown.

For $a/b < 5$, the results show agreement to a varying degree over a successively smaller region of $\xi_1 a$ as a/b decreases. The most striking feature of the curves for $a/b = 1, 2$ and 3 is that the disagreement between the two solutions becomes excessively large for values of $\xi_1 a$ narrowly distributed around a few discrete points. The rapid change in the time lag between the occurrence of $\tau_\theta^{(0)}$ and $\tau_x^{(0)}$ as a function of $\xi_1 a$ is also obvious in the vicinity of these discrete values. It is possible to relate these occurrences to the presence of an edge resonance. This will be done when the amplitudes of the waves generated at the boundary of the cavity are studied.

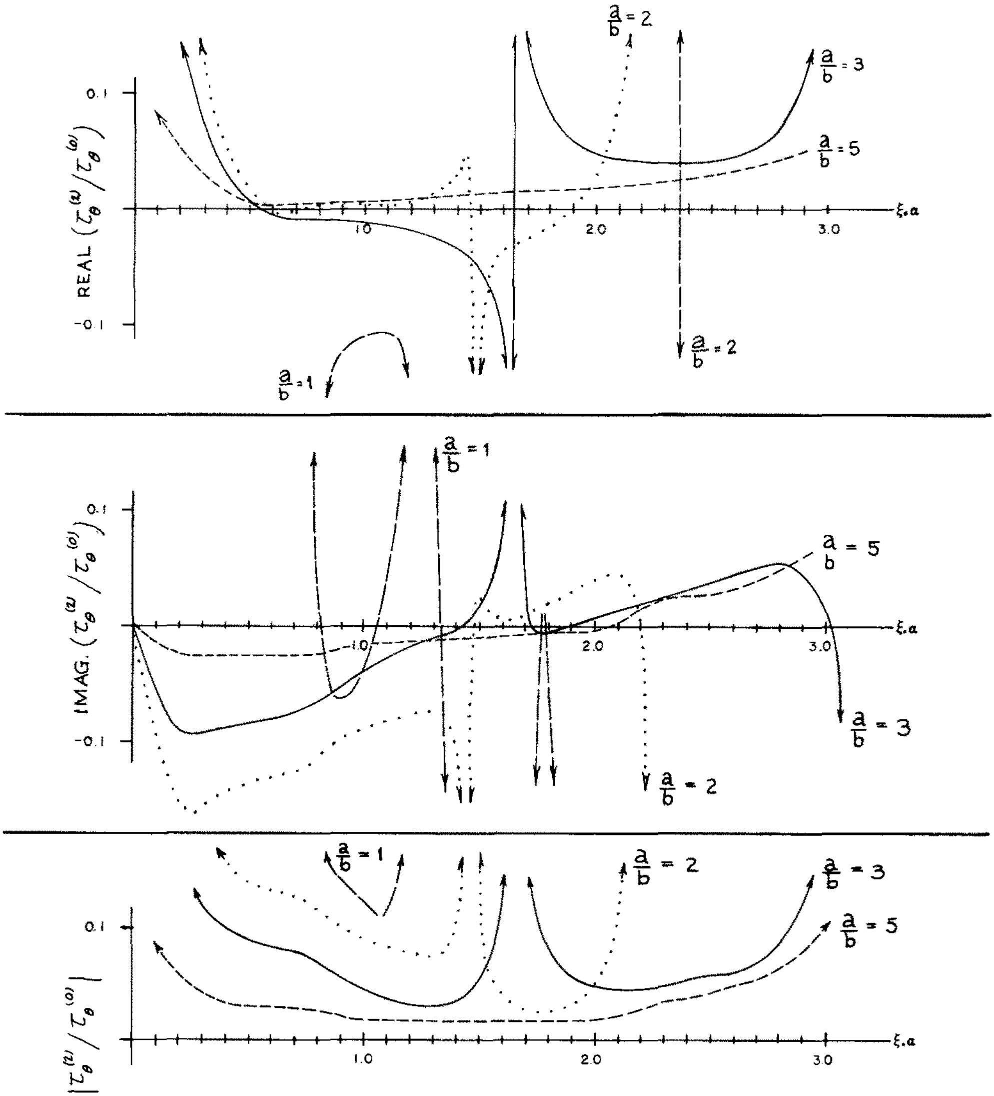

FIG. 2. Ratio of $\tau_\theta^{(2)}/\tau_\theta^{(0)}$, $\theta = 0$. (a) real part, (b) imaginary part, (c) magnitude.

It might be noted that the apparent discontinuities or infinities in the various graphs appear simply because of the scales chosen for the ordinate. The stress concentration factors are continuous and bounded functions of $\xi_1 a$; however, a greatly enlarged vertical scale would be required to show this.

Figure 2 relates $\tau_\theta^{(2)}/\tau_\theta^{(0)}$ to $\xi_1 a$ at $\theta = 0$ for various values of a/b. $\tau_\theta^{(2)}/\tau_\theta^{(0)}$ gives a measure of the departure of the stress distribution across the plate thickness from a uniform

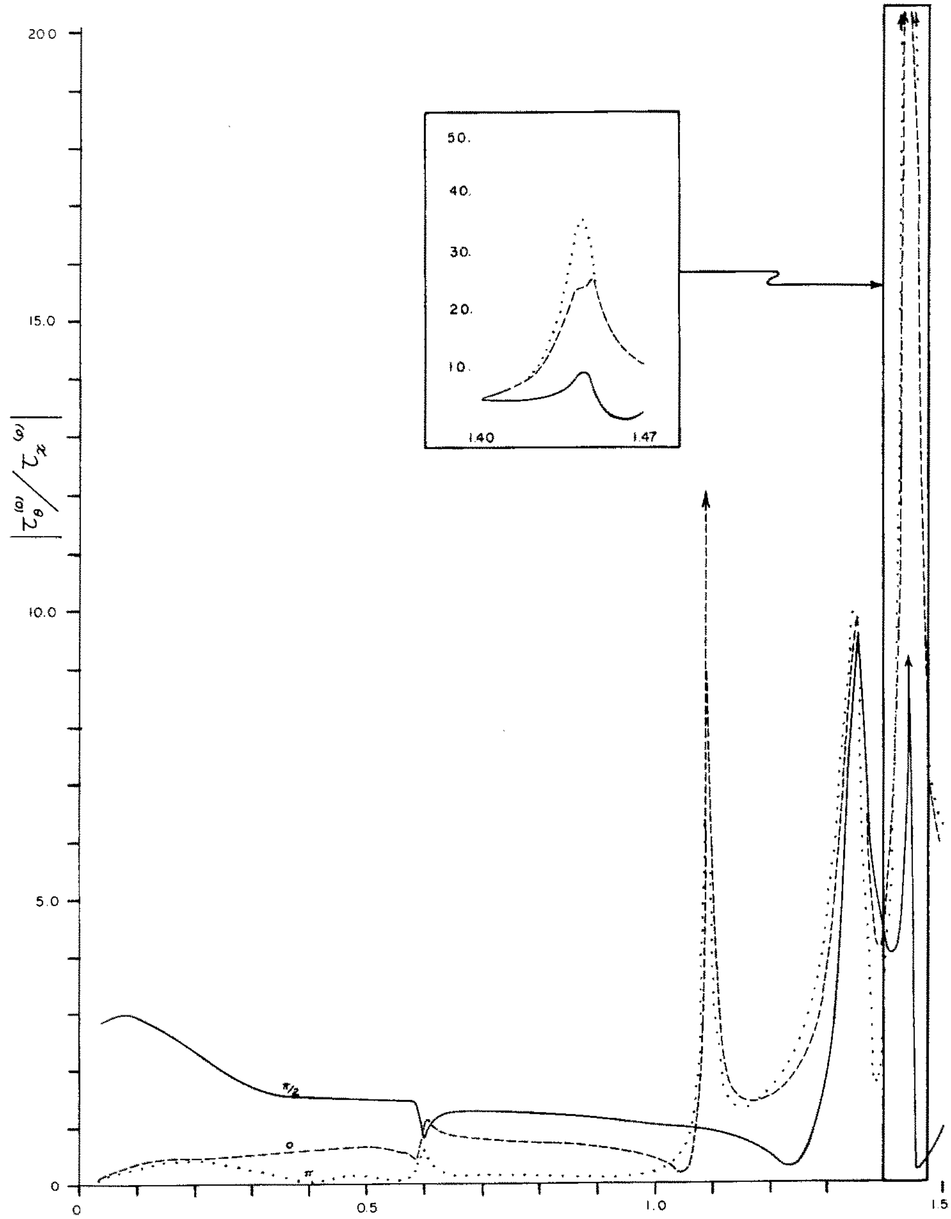

FIG. 3. Stress concentration factors for $a/b = 3$.

distribution, normalized with respect to the average stress. For $a/b = 5$, the results show this departure to be quite small over the entire range of values for $\xi_1 a$ with increases occurring at both ends of the range. The increase that occurs as $\xi_1 a$ approaches zero does so because $\tau_\theta^{(0)}$ becomes small rather than because of an increase in the non-uniform variation of stress across the thickness. For $a/b < 5$, the magnitude of $\tau_\theta^{(2)}/\tau_\theta^{(0)}$, in general, increases for a fixed value of $\xi_1 a$ as a/b becomes smaller. This conclusion is obviously not valid in the vicinity of an $\xi_1 a$ which corresponds to an edge resonance for a given a/b.

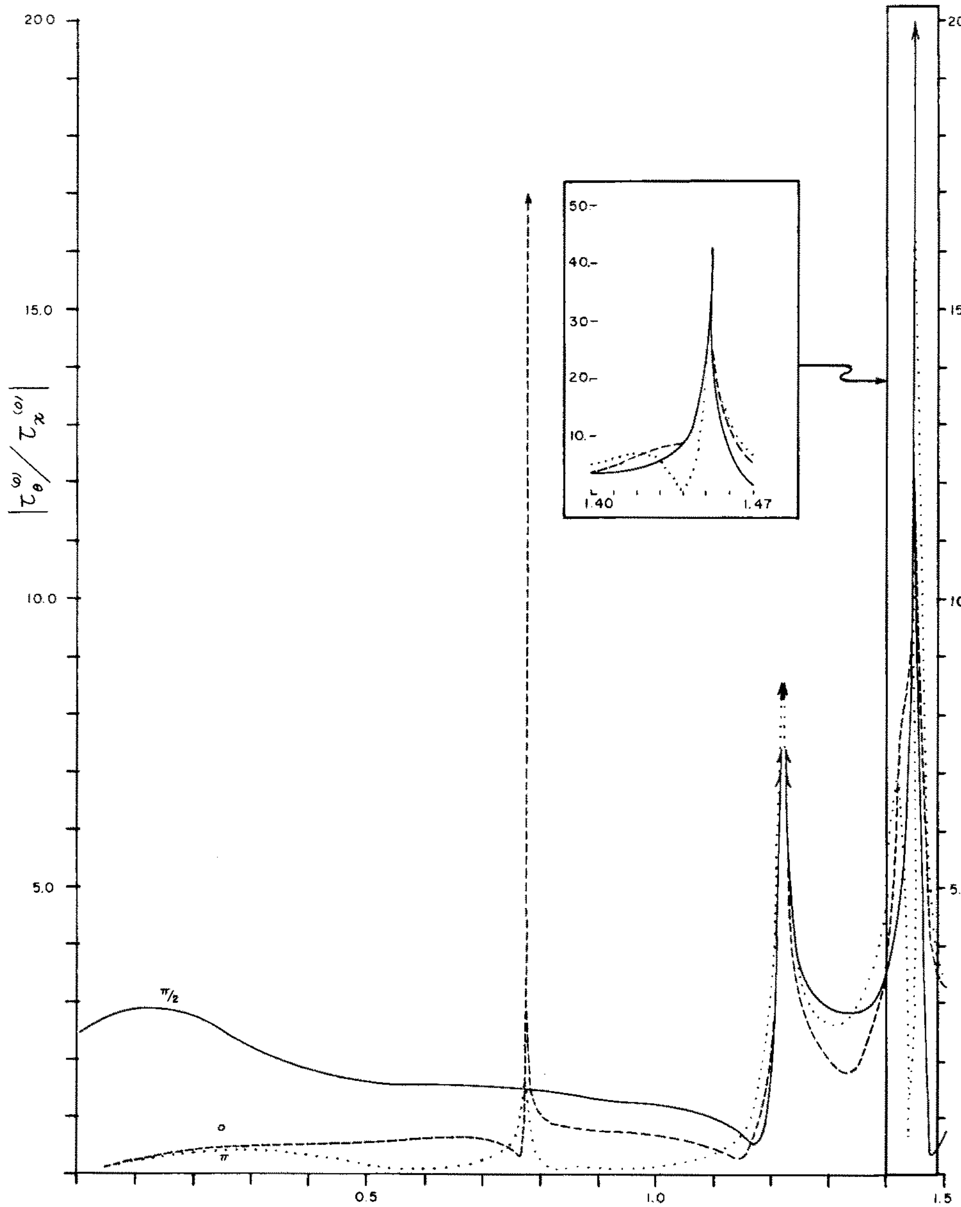

FIG. 4. Stress concentration factors for $a/b = 2$.

In such a region the magnitude of $\tau_\theta^{(2)}/\tau_\theta^{(0)}$ increases markedly for that value of a/b which corresponds to the resonance. Both of these results agree with intuition and the knowledge that the non-propagating plate waves give rise to a non-uniform stress distribution. It is to be expected that such waves will play a greater role for a given value of $\xi_1 a$ as a/b becomes smaller. In the vicinity of an edge resonance, the non-propagating waves are expected to play an even larger role.

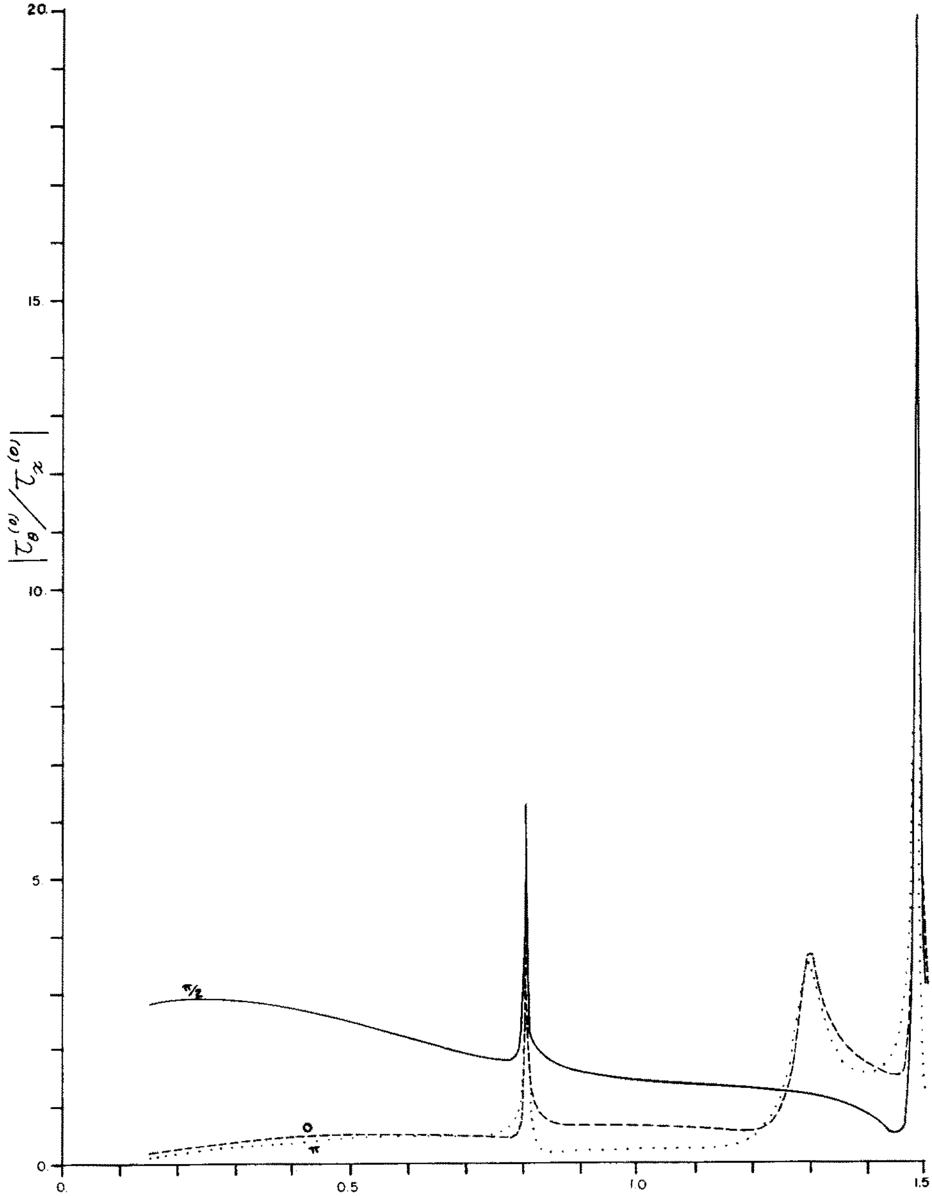

FIG. 5. Stress concentration factors for $a/b = 1$.

Figures 3–5 show the relationship between the stress concentration factors and non-dimensionalized frequency of the incident wave for several values of a/b at several positions on the boundary of the cavity. In these plots only the magnitudes are shown and the results are what should be expected from knowledge of Pao's results except for frequencies which are neighboring to a few discrete values. Table 1 lists the frequencies at which the stress concentration factors peak and the peak values obtained for these stress concentration factors. It should be remarked that in some instances it is difficult to obtain an accurate number for a maximum value of a stress concentration factor since such a factor is changing very rapidly with Ω in the vicinity of the peak, and it is searched for by incrementing Ω. An extremely small increment would have to be chosen to obtain an accurate value.

TABLE 1. MAXIMUM VALUES FOR STRESS CONCENTRATION FACTORS AND FREQUENCIES AT WHICH THESE VALUES OCCUR

Table 1a. $a/b = 3$

Ω \ θ	0	$\pi/2$	π
0·60	1	1	0·5
1·09	12	—	12
1·35	10	10	10
1·444	24	—	36
1·446	25	9	36

Table 1b. $a/b = 2$

Ω \ θ	0	$\pi/2$	π
0·78	5	—	4
1·22	9	7	7
1·42	8	—	7
1·453	43	40	41

Table 1c. $a/b = 1$

Ω \ θ	0	$\pi/2$	π
0·81	5	6	5
1·30	4	—	4
1·483	40	38	40

Such a difficulty is of little importance, however, since it is beyond the scope of the study and the accuracy of the theory to predict a precise value for the stress concentration factor. All that is of interest here is to note significant changes.

The source of the sharp increases in the stress concentration factors may be seen to reside in the amplitudes of the waves generated at the boundary of the traction free cavity by the incident plane wave. These amplitudes are given by the system of algebraic equations (15). An analysis of the amplitudes for a given n, as a function of Ω, shows that they go through a sharp maximum at a value of Ω that depends on the value of n and the value of a/b. The values of Ω at which these peaks occur are given in Table 2.

TABLE 2. NON-DIMENSIONALIZED FREQUENCIES AT WHICH AMPLITUDES OF WAVES GENERATED AT BOUNDARY GO THROUGH A PEAK VALUE

a/b \ n	0	1	2	3	4
1	1·485	1·30	0·81	—	—
2	1·45	1·42	1·22	0·78	—
3	1·448	1·44	1·35	1·09	0·60

Tables 1 and 2 allow an immediate association between a peak in a stress concentration factor and a peak in the amplitudes of some waves that arise due to the interaction of the incident wave and the cavity. It also indicates the reason why the stress concentration at $\theta = \pi/2$ does not, in some cases, change significantly when those at $\theta = 0$ and $\theta = \pi$ do; the reason being that the amplitudes which are going through a maximum are those associated with waves which have a node at $\theta = \pi/2$.

Finally, to show the behavior of the amplitudes of the generated waves in the vicinity of a peak, Fig. 6 considers the case of $a/b = 3$; $n = 2$ in the vicinity of $\Omega = 1{\cdot}44$. These graphs show both the occurrence of the peak and the fact that the phase of the generated propagating wave goes through a rapid change when taken as a function of Ω in this region.

One may attempt to draw a comparison between the results obtained in this study with those previously reported by Gazis and Mindlin pertaining to the generation of extensional waves at a traction free boundary by an incident straight crested extensional wave. Their results predict an edge resonance, characterized by a maximum for the amplitudes of the non-propagating waves at a non-dimensionalized frequency, $\Omega = 1{\cdot}314$.

In the present problem, for the case of $n = 0$ the comparison appears straightforward since the boundary approaches a plane boundary as a/b becomes large and the circularly crested waves, which correspond to $n = 0$, approach straight crested waves as a/b becomes large. By taking appropriate limits it may be shown that the coefficients of the amplitudes of the generated waves in equations (15) approach the corresponding quantities given in Ref. [5] as a/b approaches infinity. Numerically, the results obtained here give as the frequencies at which peaks occur, $\Omega = 1{\cdot}483$, $1{\cdot}453$, and $1{\cdot}446$ for $a/b = 1, 2$ and 3, respectively. Thus, a gradual tendency toward $\Omega = 1{\cdot}314$ may be noted.

For $n > 0$, the comparison is not as straightforward since the generated waves are not circularly crested but rather have two or more nodal lines perpendicular to the boundary. In the limit of a/b approaching infinity, the distances between these nodal lines become infinite for any finite n and once again it may be shown that the limit will approach the straight crested waves as above. For finite a/b, however, it might be expected that the finite distances between nodes in the circumferential direction will introduce significant differences in the frequencies at which an edge resonance occurs.

The results show that for a given a/b, the frequency at which the amplitudes peak decreases with an increase in n; while for a given n, exclusive of $n = 0$ the frequency increases for an increase of a/b. Furthermore, the difference between resonant frequencies associated with any two n's becomes smaller as a/b increases. This behavior appears to strengthen the possible conjecture that resonance occurs when some additive combination of the propagation constant of a generated wave, which determines the frequency, and the inverse of the distance between nodes in the circumferential direction attains a certain value.

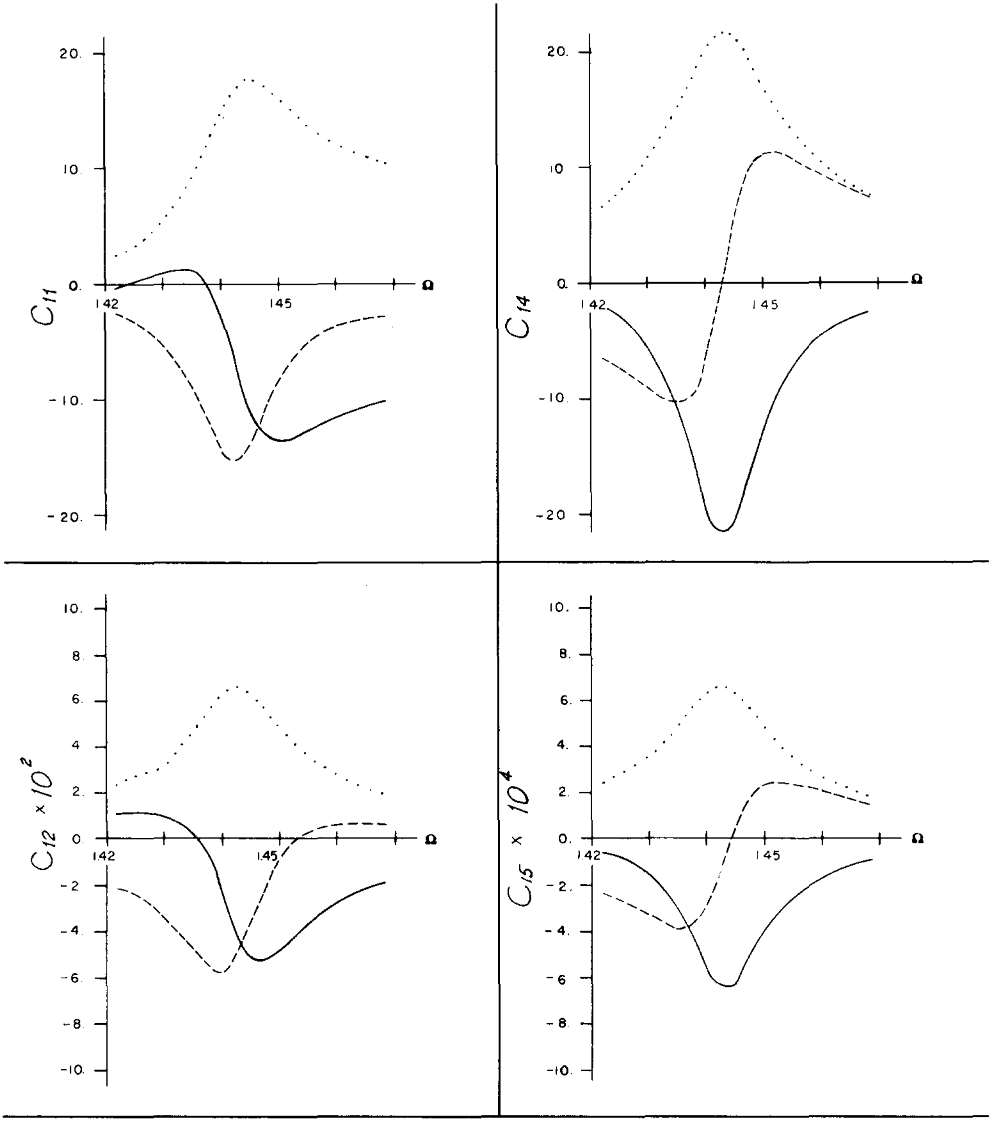

FIG. 6. Amplitudes of reflected waves as a function of non-dimensional frequency for diameter thickness ratio = 3. —— real part, --- imaginary part, ... magnitude.

CONCLUDING REMARKS

The fact that the amplitudes associated with the propagating extensional and face shearing waves that are generated at the cavity change significantly near the frequencies associated with an edge resonance indicates that the occurrence of such a resonance will have an effect on the spatial variation of the stress field in a region which is not confined to the immediate vicinity of the cavity. The effects of the propagating waves will decrease with distance from the cavity as $1/r$ whereas the effects of the non-propagating waves will decrease with distance from the cavity as approximately $\exp(-cr)$, c being some constant.

If the frequency of the incident wave is near that of a resonance corresponding to a specific n, the generated waves associated with this value of n will be of a greater amplitude than they would be for other frequencies. Hence, the spatial variation of the stress field should be influenced to a greater extent by these waves at these frequencies. Since the changes in amplitudes with frequency is great near a resonance, a rapid change of spatial variation with frequency should also occur.

Another feature that is obvious is the rapid change in phase of the generated propagating waves with frequency in the vicinity of a resonance. In the case of a transient disturbance incident upon the cavity, this rapid change in phase will result in the energy contained at frequencies in the vicinity of an edge resonance being dispersed. As reasoned by Gazis and Mindlin, this occurs since the energy contained at such frequencies is temporarily trapped in the vibration set up in the vicinity of the cavity and only gradually leaked into the propagating waves. Consistent with the result above, the energy will be leaked into those propagating waves which are associated with the same n as is the edge resonance. This temporary entrapment and leakage of energy will result in a long tail appended to a transient which is a superposition of waves of characteristic frequencies (i.e. frequencies of edge resonances) and characteristic spatial variations.

Finally, one might draw an analogy between the second order theory of extensional vibrations of plates used in this report and the three dimensional theory of materials with micro-structure presented by Mindlin [6]. As pointed out by Mindlin, the dispersion relations of the second order plate theory have their counterpart in the three dimensional dispersion relations obtained within the framework of theory governing materials with micro-structure leading one to suspect that effects found using the second order plate theory might also have their counterpart. The possibility of large values for dynamic stress concentrations occurring at select frequencies may be one of these effects. It would be one of the more interesting effects since it would represent a phenomenon that is not contained in classical elasticity theory rather than a case of adding a numerical correction to an elasticity solution.

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Абстракт—Исследуется концентрация напряжений вокруг круглого отверстия в бесконечной упругой пластинке, при переходе плоской волны растяжения, в рамках приближенной теории, учитывающей эффект сопряжения между симметрическими поперечными деформациями растяжения и симметрическими поперечными деформациями сдвига. Специальное внимание следует обратить на очень большую концентрацию напряжений, которая получается для некоторых комбинаций толщины пластинки, диаметра отверстия и случайной длины волны. Кажется, что источник этой большой концентрации напряжений, вызван наличием вибрации большой степени, которая ограничивается районом непосредственной окрестности отверстия.